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Moduli of Topological Conjugacy of Ω-stable Flows on Surfaces

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Introduction

The topological classification of structurally stable diffeomorphisms on closed manifolds has made tremendous progress in the last 50 years. A whole series of papers by such authors as S.H. Aranson, A. N. Bezdenezhnykh, V. Z. Greenes [2], [4], [6], [5], [15]; E. A. Borevich [9]; C. Bonatti, R. Langevin [8]; I.Yu. Vlasenko [40]; V.Z. Grines, S.H. Zinina, T.M. Mitryakova, O.V. Pochinka [34], [16]. The classification of arbitrary Morse-Smale diffeomorphisms on surfaces[1] required the use of the apparatus of topological Markov chains and follows from the work of C. Bonatti and R. Langevin [8] (see also [7]), where necessary and sufficient conditions of topological conjugacy are found for structurally stable diffeomorphisms with zero-dimensional basic sets.

According to the work by S. Newhouse and J. Palis [36], there is an open set of arcs that start in a Morse-Smale diffeomorphism and have the first bifurcation point in a diffeomorphism with heteroclinic tangency. The review [3] describes bifurcations of systems belonging to the boundary of the set of Morse-Smale systems, which can be divided into two parts: 1) systems with a finite set of nonwandering trajectories containing either nonhyperbolic fixed points or cycles, or trajectories of nontransversal intersection of stable and unstable manifolds of fixed points or (and) cycles, or both at the same time; 2) systems with an infinite set of nonwandering trajectories.

Obviously, the violation of the transversality condition for heteroclinic intersections of invariant manifolds of saddle points of a diffeomorphism leads to its non-roughness. Moreover, this leads to the appearance of continuous topological invariants – moduli of topological conjugacy and, hence, to the existence of a continuum of non-conjugate diffeomorphisms with the same heteroclinic intersection geometry. The term “modulus of topological conjugacy” was proposed by L.P. Shilnikov, S.V. Gonchenko and D.V. Turaev [14], [13] and corresponds to the term “moduli of stability” which is used in Western literature. Moduli of stability arise, in particular, for systems lying on the boundary of the set of Morse-Smale systems, having a finite set of nonwandering trajectories and containing trajectories of nontransversal intersection of stable and unstable manifolds of fixed points and/or cycles (see [3]).

A rigorous definition of moduli was given in the works by L.P. Shilnikov, S.V. Gonchenko and D.V. Turaev [14], [13]. Namely, let X be a topological space, $x \in X$, and let R be an equivalence relation on some neighborhood $U_x \subset X$ of x. Suppose that a continuous *locally non-constant* function $h\colon U_x \to \mathbb{R}$ is defined on U_x, that is, in any neighborhood $U_y \subset U_x$ of any point $y \in U_x$ there exists a point z such that $h(z) \neq h(y)$. We will call a function h by *an R-equivalence modulus* if the inequality $h(y) \neq h(z)$ for $y, z \in U_x$ implies that y and z are not R-equivalent. In this case, $x \in X$ is said to have *modulus* h. We say that x has (at least) m *moduli* if m independent moduli are defined on X, where *the independence of the system of moduli* $h_1, \dots, h_m$ is understood in the following sense: for any $i \in \{1, \dots, m\}$ in

[1] In this work a surface is always 2-dimensional.

any neighborhood $V_x \subset U_x$ of x there exists a point y such that $h_l(x) = h_l(y)$ for all $l \neq i$ and $h_i(x) \neq h_i(y)$. We say that x *has infinitely many moduli* if x has m moduli for any given m. Otherwise, x *has a finite number of moduli.*

If in this definition we replace $\mathbb{R}$ with a space of some functions, and replace the equality of the values of the map h with some equivalence relation of the values of the map h, then h will be called *a functional R-equivalence modulus.*

J. Palis was the first to pay attention to the existence of topological conjugacy moduli [38]. He discovered the existence of topological conjugacy moduli for systems with simple dynamics. Two-dimensional diffeomorphisms and flows with a non-rough heteroclinic trajectory, at the points of which the invariant manifolds of two different saddle fixed points have one-sided tangency, already have such moduli. Namely, if f is such a diffeomorphism (of class $C^r, r \geq 2$) that has two hyperbolic saddle fixed points σ_1 and σ_2 with eigenvalues ϱ_i, μ_i such that $|\varrho_i| < 1 < |\mu_i|,\ i = 1, 2$; moreover, $W^s_{\sigma_1}$ has a one-way tangency with $W^u_{\sigma_2}$ at points of some heteroclinic trajectory (see Fig. 1), then the parameter

$$\alpha = \frac{\ln|\varrho_2|}{\ln|\mu_1|}$$

is a modulus of topological conjugacy in the sense that diffeomorphisms f and f' with heteroclinic tangency can be conjugate only if

$$\frac{\ln|\varrho_2|}{\ln|\mu_1|} = \frac{\ln|\varrho'_2|}{\ln|\mu'_1|}.$$

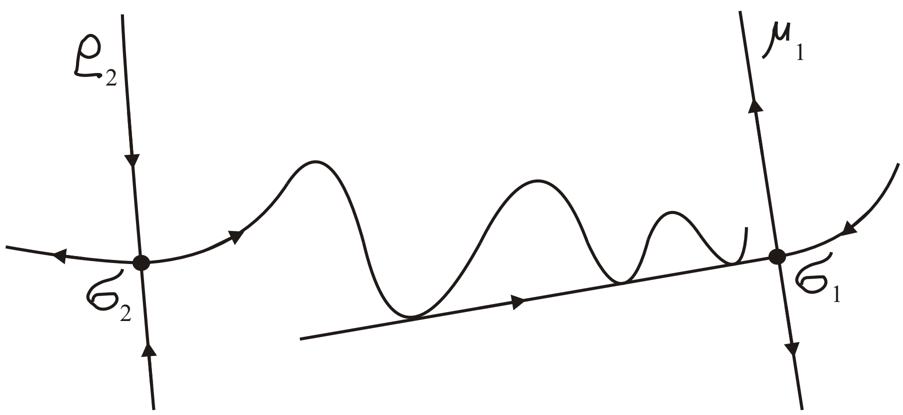

Figure 1: Tangency of saddle invariant manifolds

It follows from the above, in particular, that any diffeomorphism of a surface that admits a heteroclinic tangency has at least one topological conjugacy modulus. A significant advance in the description of the moduli of surface diffeomorphisms was the work of W. di Melu, S. van Strien [11], in which necessary and sufficient conditions were found for an Ω-stable diffeomorphism f of an orientable surface to have a finite number of moduli of topological conjugacy. Note that one of the conditions is that the length of the chain of tangent separatrices is bounded.

This study is devoted to describing the moduli of topological conjugacy of Ω-stable flows on surfaces, distinguishing among them a class of flows with a finite number of moduli, and classifying them up to topological conjugacy.

The traditional approach to the qualitative study of the dynamics of flows with a finite number of fixed points and periodic orbits on surfaces consists in identifying regions on the carrier manifold with the same asymptotic behavior of trajectories which are called *cells*. The classical combinatorial invariants of such flows based on cell selection are the Leontovich-Mayer scheme [32], [31] for flows in a bounded part of the plane, the directed Peixoto graph [39] and the Oshemkov-Sharko molecule [37] for Morse-Smale flows on arbitrary closed surfaces, the Neumann-O'Brien orbital complex [35] for the class of flows on arbitrary closed surfaces containing Ω-stable flows.

Recall that a Morse-Smale flow is called *gradient-like* if its nonwandering set does not contain periodic trajectories. Such flows have the simplest dynamics, which inspired many mathematicians to look for invariants of their topological equivalence. Under assumptions of different generality, the following invariants were obtained for the considered class of gradient-like flows: Peixoto graph (M. Peixoto) [39], modified Peixoto graph (V.Z. Grines, O.V. Pochinka) [17] , bicolor graph (X. Wang) [41], three-colour graph (A.A. Oshemkov, V.V. Sharko) [37], circular scheme (G. Fleitas) [12].

Thus, the problem of classifying gradient-like flows on surfaces from the point of view of topological equivalence has been solved in an exhaustive way. In this work, we prove that for gradient-like flows the topological equivalence classes coincide with the topological conjugacy classes. The obtained result allows us to use any invariants of their equivalence to check the topological conjugacy of gradient-like flows. In addition, for each of the above invariants, an *efficient algorithm* (its running time depends polynomially on the input data) is constructed to distinguish the topological equivalence of gradient-like flows.

Obviously, each limit cycle generates a topological conjugacy modulus equal to the period of the cycle. Therefore, for the topological conjugacy of Morse-Smale flows, the available topological equivalence invariants are clearly not sufficient. In addition, in this paper, the surprising fact of the presence of an infinite number of topological conjugacy classes in one class of topological equivalence of the Morse-Smale flow is established. This effect is related to the uniqueness of the invariant foliation in a neighborhood of any periodic orbit. It is proved that the criterion for the finiteness of the number of moduli of topological conjugacy of a Morse-Smale flow on a surface is the absence of trajectories going from one limit cycle to another one. For the class of Morse-Smale flows on surfaces with a finite number of moduli, their topological classification is also obtained up to topological conjugacy, based on the Oshemkov-Sharko molecule.

Another source of moduli for Ω-stable systems on surfaces is the existence of tangent saddle invariant manifolds, i.e. *connections* (see Fig. 2), which was discovered by J. Palis in [38]. As follows from the above results of S. van Strien and W. de Melu, one of the conditions for the finiteness of the number of moduli for a diffeomorphism is the limitation of the length of the chain of saddles with tangent saddle manifolds, it should not exceed three. In this work, we prove that in the case of a flow there is no such restriction on the length of the saddle chains of the flow. A complete invariant of topological equivalence of Ω-stable flows, an equipped graph, is also introduced; for such graphs, a polynomial algorithm for distinguishing their isomorphism is constructed.

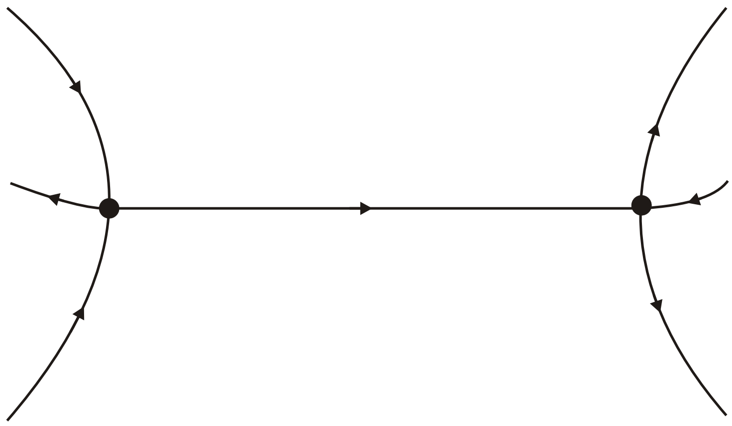

Figure 2: A connection

1 The results

Let M be a smooth closed n-manifold with a metric d. *A smooth flow* on M is a smooth map $\phi\colon M \times \mathbb{R} \to M$ with group properties:
1) $\phi(x, 0) = x \ \forall x \in M$;
2) $\phi(\phi(x, t), s) = \phi(x, t + s) \ \forall x \in M, \forall s, t \in \mathbb{R}$.

In what follows, we will use the notation $\phi^t(x) = \phi(x, t)$, $x \in M$, $t \in \mathbb{R}$. Note that for a fixed $t \in \mathbb{R}$ the map $\phi^t\colon M \to M$ is a diffeomorphism (see, for example, [19]), so the flow is also called the one-parameter group of diffeomorphisms acting on the manifold M.

The *trajectory* or *orbit* of a point $x \in M$ is called the set $\mathcal{O}_x = \{\phi^t(x), t \in \mathbb{R}\}$. Any flow trajectory either consists of a single point, in which case this point is called *fixed*, or is homeomorphic to a circle, in which case any point of the trajectory is called *periodic*, or is an injectively immersed line. It is assumed that all flow trajectories other than a fixed point are oriented in accordance with the increase in the parameter t. Each flow $\phi^t\colon M \to M$ is associated with a vector field tangent to the trajectories of the flow

$$\dot{x} = \frac{\partial \phi(x, t)}{\partial t}|_{t=0} = F(x).$$

Flows f^t, $f'^t\colon M \to M$ on a manifold M are said to be *topologically equivalent* if there exists a homeomorphism $h\colon M \to M$ mapping the trajectories of the flow f^t to trajectories of the flow f'^t with preservation of the direction of movement along the trajectories. Two flows are said to be *topologically conjugate* if the condition $hf^t = f'^t h$, $t \in \mathbb{R}$ is satisfied, which means that h maps trajectories to trajectories, preserving not only the direction, but also the time of movement along the trajectories.

Let $p \in M$ be a fixed point of the flow $\phi^t \colon M \to M$. *Stable and unstable*, respectively, the manifold of a fixed point p are the sets

$$W_p^s = \{x \in M : d(p, \phi^t(x)) \to 0 \text{ при } t \to +\infty\} \text{ и}$$
$$W_p^u = \{x \in M : d(p, \phi^t(x)) \to 0 \text{ при } t \to -\infty\}.$$

The *stable (unstable) separatrix* of a fixed point p is a connected component of the set $W_p^s \setminus p$ ($W_p^u \setminus p$).

In addition to the tangent vector field $\dot{x} = F(x)$, the fixed point of the flow is associated with a *linearized vector field*

$$\dot{x} = A(x - p),$$

where A is the matrix of partial derivatives of the map $F(x)$ at the point p (Jacobi matrix). A fixed point p of the flow ϕ^t is called *hyperbolic* if the eigenvalues of the matrix A do not have zero real parts.

Moreover, a fixed point p of the diffeomorphism $f \colon M \to M$ is called *hyperbolic* if the partial derivative matrix of the mapping $f(x)$ at the point p (Jacobi matrix) has no eigenvalues in module equal to one.

Let $\mathfrak{c}$ be a closed trajectory of the flow $\phi^t \colon M \to M$. *Stable and unstable manifolds*[2] respectively of the closed trajectory $\mathfrak{c}$ are the sets

$$W_{\mathfrak{c}}^s = \{x \in M : \min_{p \in \mathfrak{c}} d(p, \phi^t(x)) \to 0 \text{ if } t \to +\infty\} \text{ and}$$
$$W_{\mathfrak{c}}^u = \{x \in M : \min_{p \in \mathfrak{c}} d(p, \phi^t(x)) \to 0 \text{ if } t \to -\infty\}.$$

Let $p \in \mathfrak{c}$, and Σ_p be an $(n-1)$-dimensional disk transversal at p to a vector tangent to the periodic trajectory, called the *Poincaré secant.* Then in some neighborhood $V_p \subset \Sigma_p$ of the point p for each point $x \in V_p$ there exists a value $\tau_x > 0$ such that $\phi^{\tau_x}(x) \in \Sigma_p$ and $\phi^t(x) \notin \Sigma_p$ for any $0 < t < \tau_x$. The map $f \colon V_p \to \Sigma_p$ defined by the formula $f(x) = \phi^{\tau_x}(x)$, $x \in V_p$ is called the *succession mapping* or the Poincaré map.

The point p is a fixed point of the sequence map. A periodic trajectory $\mathfrak{c}$ is called *hyperbolic* if the point p is a hyperbolic fixed point of the Poincaré map $f \colon V_p \to \Sigma_p$.

A point $x \in M$ is called a *wandering point* of a flow $\phi^t \colon M \to M$ if there exists an open neighborhood U_x of x such that $\phi^t(U_x) \cap U_x = \varnothing$ for all $t > 1$. Otherwise the point x is called *non-wandering*, the set of all non-wandering points of the flow ϕ^t is called its *non-wandering set* and denoted by Ω_{ϕ^t}.

[2]See [42], The Stable Manifold Theorem.

A flow $\phi^t\colon M \to M$ is called *Ω-stable* if there exists a neighborhood $U(\phi^t)$ of the flow ϕ^t in the space $C^1(M \times \mathbb{R}, M)$ with C^1-topology such that if $\phi'^t \in U(\phi^t)$ then the flows $\phi^t|_{\Omega_{\phi^t}}$ and $\phi'^t|_{\Omega_{\phi'^t}}$ are topologically equivalent.

A flow $\phi^t\colon M \to M$ is called *structurally stable* if there exists a neighborhood $U(\phi^t)$ of the flow ϕ^t in the space $C^1(M \times \mathbb{R}, M)$ with C^1-topology such that if $\phi'^t \in U(\phi^t)$ then the flows ϕ^t and ϕ'^t are topologically equivalent.

A flow is called a *Morse-Smale flow* if its nonwandering set consists of a finite number of hyperbolic fixed points and a finite number of hyperbolic periodic orbits whose stable and unstable manifolds intersect transversally. A Morse-Smale flow without periodic orbits is called a *gradient-like* flow.

As a part of the study, the following results were obtained on the topological conjugacy of Ω-stable flows on surfaces.

Chapter 2 considers gradient-like flows on surfaces and proves that for such flows the classifications coincide up to topological equivalence and topological conjugacy. For the main topological invariants of flows of this class, efficient algorithms for distinguishing them are found.

Namely, consider a gradient-like flow f^t given on a closed surface S. The first result of the chapter says that the topological invariants describing the topological equivalence classes of gradient-like flows on surfaces are also suitable for classification up to topological conjugacy.

Theorem 1 ([21]*, Theorem 7; [28]*, Theorem 2.1; [30]*, Theorem 1) *If two gradient-like flows on a closed surface are topologically equivalent, then they are topologically conjugate.*

Thus, the classes of topologically equivalent and the classes of topologically conjugate gradient-like flows on surfaces coincide. In most cases, the invariants describing these classes are equipped graphs. Two equipped graphs are said to be *isomorphic* if there is a one-to-one correspondence that maps vertices and edges of one graph to vertices and edges of the other graph with preservation of equippings. An algorithm for distinguishing graph isomorphism in some class of graphs is called *efficient or polynomial* if its implementation time is limited by a polynomial of the length of the input information (the number of vertices, edges, and graph framing parameters). This definition of algorithm efficiency goes back to A. Cobham [10]. Since for arbitrary graphs the problem of the existence of an efficient distinguishing algorithm (the NP-completeness problem) is open, the efficiency is the standard of intractability of such a problem [18].

Consider the set

$$\tilde{S} = S \setminus \bigcup_{\sigma \in \Omega^1_{f^t}} \left(cl(W^u_\sigma) \cup cl(W^s_\sigma) \right).$$

The closure of any of its connected components is called a *cell*.

Let Γ_{f^t} be a directed flow graph f^t such that the vertices of the graph Γ_{f^t} correspond to fixed points of the flow f^t, and the edges correspond to saddle separatrices. Let's equip the graph Γ_{f^t} with *distinguishing sets* i.e. subgraphs corresponding to the cell boundaries. As a result, we get *Peixoto graph* $\Gamma^P_{f^t}$. Such a graph is a complete topological invariant for gradient-like flows on arbitrary surfaces (see Fig. 3). .

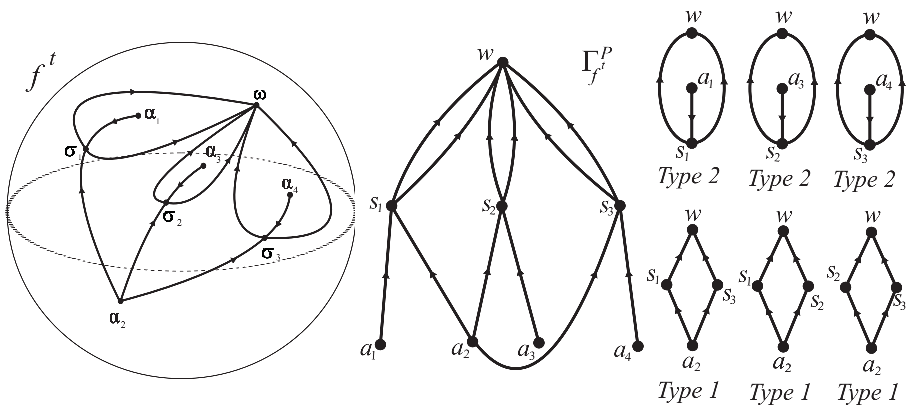

Figure 3: Gradient-like flow f^t on the sphere S and its Peixoto graph $\Gamma^P_{f^t}$

Theorem 2 ([21]*, Theorem 1; [28]*, Theorem 3.1) *Let f^t and f'^t be gradient-like flows defined on a surface S of genus g, and $\Gamma^P_{f^t}$, $\Gamma^P_{f'^t}$ are their n-vertex Peixoto graphs. Then the isomorphism of the graphs $\Gamma^P_{f^t}$ and $\Gamma^P_{f'^t}$ can be checked in time $O\left(n^{O(g)}\right)$ for $g>0$ and in time $O(n)$ for $g=0$.*

In 2011 V. Z. Grines and O. V. Pochinka [17] modified Peixoto graph. Namely, instead of distinguishing sets, they equipped the directed Peixoto graph Γ_{f^t} with orders of edges (consistent with the embeddings of saddle separatrices in the supporting surface) incident to the vertices corresponding to sinks. The isomorphism class of the thus obtained *modified Peixoto graph* $\Gamma^{GP}_{f^t}$ is also a complete equivalence invariant of gradient-like flows on arbitrary surfaces.

Theorem 3 ([21]*, Theorem 2; [28]*, Theorem 3.2) *Let f^t, f'^t be gradient-like flows on a surface S of genus g, and $\Gamma^{GP}_{f^t}$, $\Gamma^{GP}_{f'^t}$ are their modified n-vertex Peixoto graphs. Then the graph isomorphism $\Gamma^{GP}_{f^t}$ and $\Gamma^{GP}_{f'^t}$ can be checked in $O\left(n^{O(g)}\right)$ if $g>0$, and in time $O(n)$ if $g=0$.*

The next invariant for which the distinguishing algorithm is constructed is the *Wang graph* [41]. The Wang graph for the flow f^t on an orientable surface is the graph dual to the Peixoto graph: the vertices of the Wang graph $\Gamma^W_{f^t}$ correspond to the cells of the flow f^t, its edges correspond to the saddle separatrices and connect the vertices corresponding to the cells bordering separatrices along the corresponding edges. The edge is coloured in the colour u if it corresponds to an unstable saddle separatrix, and in the colour s if it corresponds to a stable saddle separatrix. Moreover, if any saddle separatrix lies in the interior of the closure of some cell, then this cell and this separatrix correspond to a vertex of a graph with a loop. That is, each vertex has a valence 4, if we count the loop as two 'edges'. The set of these four edges is divided into pairs, each of which includes one edge corresponding to a stable separatrix and one edge corresponding to an unstable separatrix, adjoining each other at the boundary of the corresponding cell vertex. Such pairs are denoted by an arc that intersects both edges of the pair (see Fig. 4).

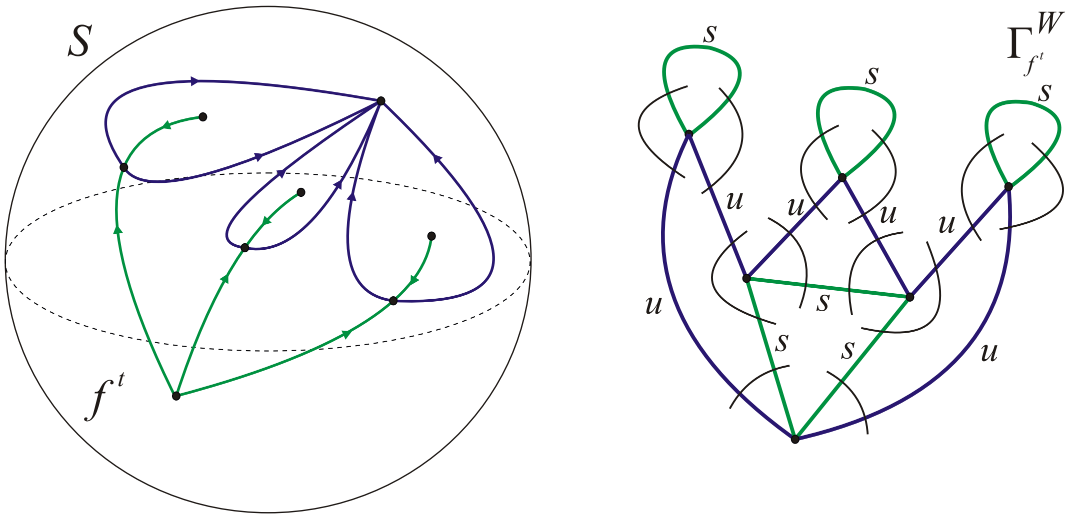

Figure 4: A flow f^t from G onto a surface S and its Wang graph $\Gamma^W_{f^t}$

Theorem 4 ([21]*, Theorem 3; [28]*, Theorem 3.3) *Let f^t, f'^t be gradient-like flows on an orientable surface S of genus g, and $\Gamma^W_{f^t}$, $\Gamma^W_{f'^t}$ are their n-vertex Wang graphs. Then the isomorphism of the graphs $\Gamma^W_{f^t}$ and $\Gamma^W_{f'^t}$ is checked in time $O\left(n^{O(g)}\right)$ if $g > 0$ and in time $O(n)$ if $g = 0$.*

A gradient-like flow $f^t\colon S \to S$ is called *polar* if its nonwandering set contains exactly one source and exactly one sink. The *Fleitas graph* or *Fleitas circular scheme* $\Gamma^F_{f^t}$ for such a flow f^t is constructed as follows. Let us choose around the source (the only one, due to the polarity of the flows) a circle S transversal to the trajectories of the flow f^t in the source basin. Denote by D the disk that this circle bounds in the source basin (i.e. 2-dimensional invariant manifold). Assign *labels* to all points of intersection of the circle S with saddle separatrices so that the points of intersection with separatrices of the same saddle have the same labels. Each pair of points with the same labels is assigned with *spin*, that is, the sign $+(-)$, if the union of the disk D with the tubular neighborhood of the stable manifold of the saddle point that intersects the circle $\mathcal{S}$ at the given pair of points is an annulus (Möbius band) (see Fig. 5). Actually, the Fleitas graph is a circle $\mathcal{S}$ with intersection points equipped with assigned labels and spins, in which the intersection points are vertices, and the arcs of the circle $\mathcal{S}$ connecting these vertices are edges.

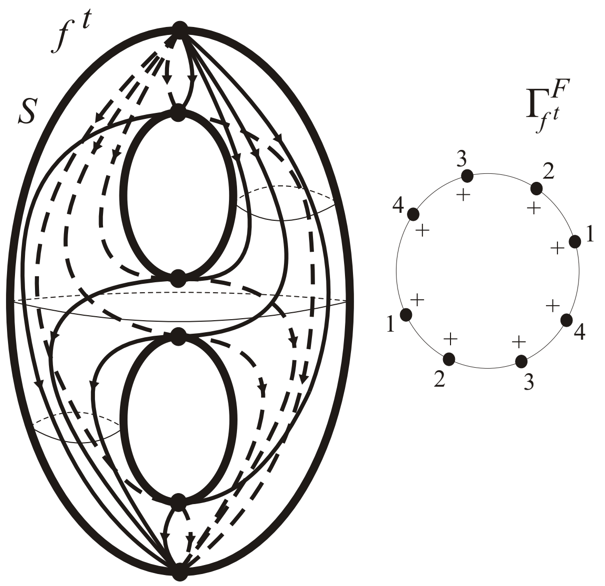

Figure 5: Polar flow f^t and its Flutas graph $\Gamma^F_{f^t}$

Theorem 5 ([21]*, Theorem 5; [28]*, Theorem 3.4) *Let f^t and f'^t be polar flows on a surface S of genus g, and $\Gamma^F_{f^t}$, $\Gamma^F_{f'^t}$ are their n-vertex Fleitas graphs. Then the isomorphism of the graphs $\Gamma^F_{f^t}$ and $\Gamma^F_{f'^t}$ is checked in time $O(n^{O(g)})$, if $g > 0$, and in time $O(n)$, if $g = 0$.*

The last considered invariant is intended again for arbitrary gradient-like flows on surfaces. Denote by J_{f^t} the set of all cells of the flow f^t. We choose one trajectory θ_J (*t-curve*) in each cell $J \in J_{f^t}$. Let $\mathcal{T} = \bigcup\limits_{J \subset \tilde{S}} \theta_J$, $\bar{S} = \tilde{S} \backslash \mathcal{T}$. We call by *$u$-curves* unstable saddle separatrices and by *s-curves* stable saddle separatrices. It follows from [39] that each connected component Δ of $\bar{S}$ is a curvilinear triangle bounded by one s-, one u- and one t-curve, so we will call Δ by a *triangular region*. Denote by Δ_{f^t} the set of all triangular regions of the flow f^t.

The three-color graph $\Gamma^{OS}_{f^t}$ by Oshemkov-Sharko from [37] corresponding to the gradient-like flow f^t is constructed as follows (see Fig. 6):

1) the vertices of the graph $\Gamma^{OS}_{f^t}$ one-to-one correspond to triangular regions;

2) two vertices of the graph are incident to an edge of color s, t, u, if the polygonal regions corresponding to these vertices have a common s-, t-, or u-side, and a one-to-one correspondence is established between this edge and an s, t, or u-curve, respectively.

Theorem 6 ([21]*, Theorem 4; [28]*, Theorem 3.5) *Let f^t, f'^t be gradient-like flows defined on surfaces of genus g, and $\Gamma^{OS}_{f^t}$, $\Gamma^{OS}_{f'^t}$ – their n-vertex three-color graphs. Then the graph isomorphism $\Gamma^{OS}_{f^t}$ and $\Gamma^{OS}_{f'^t}$ are checked in $O\left(n^{O(g)}\right)$ for $g > 0$ and in time $O(n)$ for $g = 0$.*

In Chapter 3 a finiteness criterion for the number of moduli of Morse-Smale flows on surfaces was established and a classification of such flows in the sense of topological

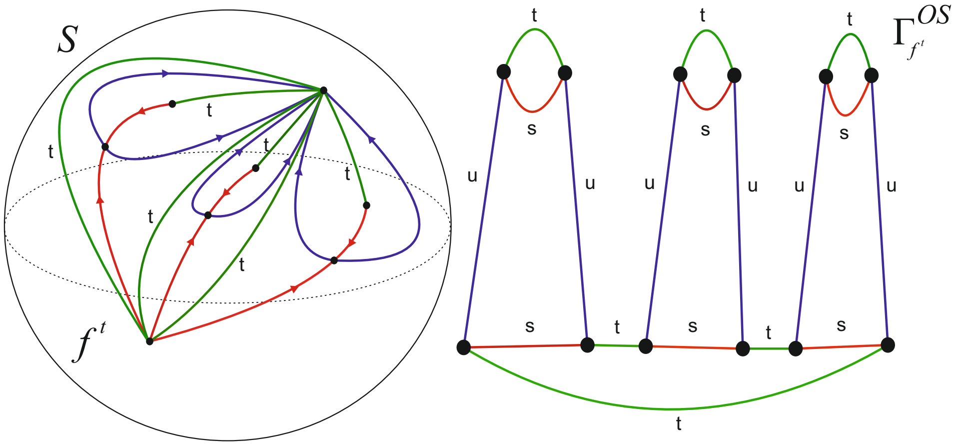

Figure 6: Phase portrait of some gradient-like flow and its three-color graph

conjugacy was obtained.

Theorem 7 ([25]*, Lemma 1; [24]*, Theorem 5.1) *If a Morse-Smale flow has an unstable limit cycle in which an unstable manifold intersects with a stable manifold of some stable limit cycle, then the flow has a functional modulus of topological conjugacy that generates an infinite number of numerical moduli of topological conjugacy.*

Consider a Morse-Smale flow ϕ^t given on a closed surface S.

Let Ω_i be the periodic orbit of ϕ^t, $K_i = W^u_{\Omega_i}$ for the repelling cycle Ω_i and $K_i = W^s_{\Omega_i}$ for the attracting cycle Ω_i, respectively.

Lemma 3.2 ([24]*, Lemma 4.1) *There is a unique ϕ^t-invariant one-dimensional foliation Ξ_i on K_i whose fibers ξ_i are secant for the trajectories of the flow $\phi^t|_{K_i}$, and*

$$\phi^{T_i}(z) \in \xi_i,\ \phi^t(z) \notin \xi_i \ if\ 0 < t < T_i,\ \ if z \in \xi_i.$$

Such a foliation, uniqueness of which is indicated by Lemma 3.1, arises from Lyapunov's work [33] and was used in the proof of Andronov-Vitt Theorem on the Lyapunov stability of a periodic trajectory [1], however, in the mentioned papers for such a foliation smoothness is required.

Further, the class of Morse-Smale flows on surfaces with a finite number of topological conjugacy moduli is distinguished.

Theorem 8 ([25]*, Theorem 1) *A Morse-Smale flow ϕ^t on a surface S has a finite number of moduli if and only if ϕ^t does not have an unstable limit cycle whose unstable manifold intersects the stable manifold of some stable limit cycle.*

Further, it is established that each topological conjugacy class of a Morse-Smale flow ϕ^t with a finite number of moduli corresponds one-to-one with an isomorphism class of some graph. To construct such a graph, a neighborhood around each limit cycle is chosen with boundary connected components that are transversal to trajectories. These boundary components divide the surface into elementary regions. Each elementary region corresponds to a vertex of the graph, and the boundary components correspond to edges directed in accordance with the direction of the trajectories intersecting the boundary component.

All graph vertices are divided into 3 types:

– $\mathcal{A}$-vertex corresponding to an elementary region containing only a single node point;

– $\mathcal{L}$-vertex corresponding to the elementary region, containing only a single limit cycle from the non-negative set;

– $\mathcal{M}$-vertex containing at least one saddle point.

Such a graph – let us denote it by Υ_{ϕ^t} – needs additional information to be a topological invariant. Therefore $\mathcal{M}$-vertex of graph Υ_{ϕ^t} is equipped with a three-color graph. Exactly, consider some $\mathcal{M}$-region which is either a 2-manifold with boundary or a closed surface. In the first case, we glue the union D of unconnected 2-disks to the boundary to get a closed surface M, in the second case we also call the already existing surface M and put $D = \varnothing$. Let us continue the flow $\phi^t|_{\mathcal{M}}$ to a gradient-like flow $f^t\colon M \to M$ such that f^t matches ϕ^t outside D, and Ω_{f^t} has exactly one fixed point (sink or source) in each connected component of set D. Assume

$$\Gamma_{\mathcal{M}} = \Gamma_{f^t}^{OS}.$$

Due to the embedding of the three-color graph $\Gamma_{\mathcal{M}}$ into the surface M, we can induce oriented boundaries of the region $\mathcal{M}$ to the graph cycle $\Gamma_{\mathcal{M}}$ corresponding to the node point lying on the disk, which replaced the $\mathcal{L}$-region. Such oriented cycles $\tau_{\mathcal{M},\mathcal{L}}$ and $\tau_{\mathcal{L},\mathcal{M}}$ we equip the edges of $\mathcal{ML}$ and $\mathcal{LM}$ of the graph Υ_{ϕ^t} respectively.

We denote the obtained equipped graph by $\Upsilon^*_{\phi^t}$. According to [37], the graph $\Upsilon^*_{\phi^t}$ is a complete Morse-Smale flow topological invariant with finite number of moduli with exact topological equivalence (see figure 7).

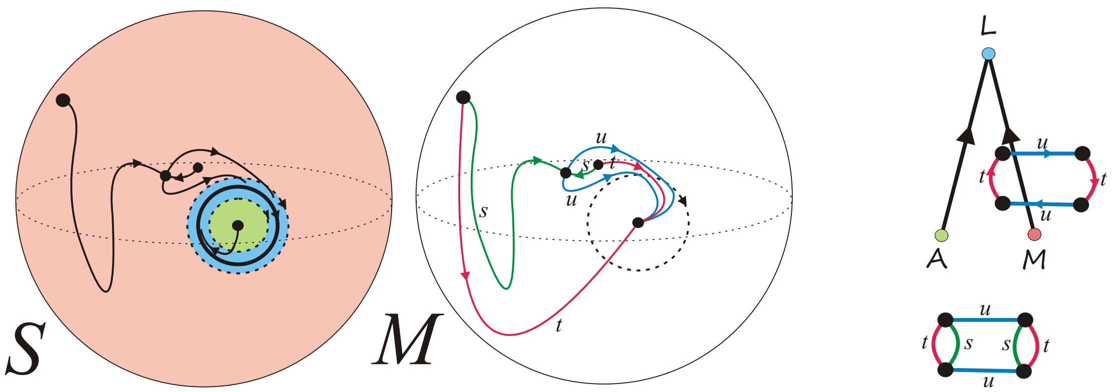

Figure 7: Morse-Smale flow phase portrait and its equipped graph

Now equip each $\mathcal{L}$-vertex of the graph with the period of the corresponding limit cycle. We denote such an equipped graph by $\Upsilon^{**}_{\phi^t}$.

Theorem 9 ([25]*, Theorem 2) *Morse-Smale flows ϕ^t, ϕ'^t without intersections of invariant manifolds of different limit cycles are topologically conjugate if and only if their equipped graphs $\Upsilon^{**}_{\phi^t}$ and $\Upsilon^{**}_{\phi'^t}$ are isomorphic.*

In Chapter 4 a classification of Ω-stable flows on surfaces is obtained up to topological equivalence. The realization of invariants by standard flows on surfaces was performed and efficient algorithms to distinguish isomorphisms were obtained. It was found that flows with arbitrarily long chain of bundles, in contrast to diffeomorphisms, have a finite number of moduli.

First, consider an Ω-stable flow f^t without limit cycles on a closed surface S. Such a flow mutually unambiguously corresponds to the four-color graph Γ_{f^t}, generalizing the three-color graph. Namely, the work proves that each component of the connectivity of the complement of the surface up to the closure of the invariant manifolds of all saddle points is *a polygon region* whose boundary, besides $s-$, $t-$, $u-$ curves can also include any finite number of c-curves which are connections. Let us orient the boundary of the polygonal region according to the positive direction of the t-curve. Denote by Δ_{f^t} the set of polygon regions of flow f^t. Let us match the flow f^t with a four-color graph as follows (see Fig. 8):

1) the vertices of the graph Γ_{f^t} correspond unambiguously to polygonal regions of the set Δ_{f^t} of the flow f^t;

2) two vertices of a graph are incident to an edge of color s, t, u or c, if the polygonal regions corresponding to these vertices contain a common s-, t-, u- or c-curve in their closures;

3) if there is more than one c-edge going from some vertex of the graph Γ_{f^t}, the c-edges are considered ordered according to the passage of the corresponding separatrices in the direction of the t-curve.

Theorem 10 ([20]*, Theorem 1; [22]*, Theorem 1.1; [26]*, Theorem 3.1) *Ω-stable flows f^t and f'^t without limit cycles are topologically equivalent if and only if their four-color graphs Γ_{f^t} and $\Gamma_{f'^t}$ are isomorphic.*

The work also defines admissible abstract four-color Γ graphs and constructively proves the following theorem.

Theorem 11 ([20]*, Theorem 3) *For any admissible graph Γ there exists a Ω-stable flow f^t without limit cycles, given on a closed surface S, whose graph is isomorphic to Γ, besides*

i) The Eulerian characteristic of the surface S is calculated by the formula

$$\chi(S) = \nu_0 - \nu_1 + \nu_2, \qquad (1)$$

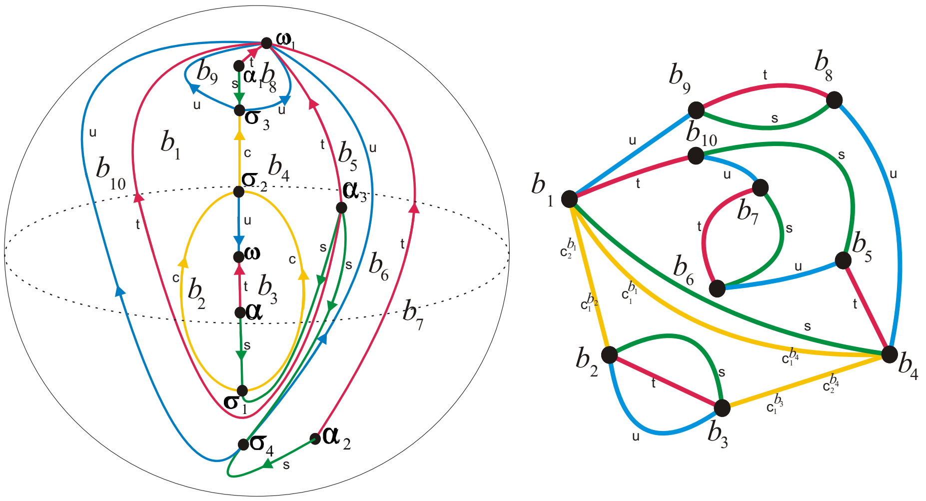

Figure 8: Phase portrait of some Ω-stable flow without limit cycles and its four-color graph

where ν_0, ν_1 and ν_2 – the number of all tu-, c- and st-cycles of the graph Γ respectively;

ii) The surface S is unorientable if and only if the graph Γ contains at least one cycle of odd length.

The paper also established an efficient algorithm for distinguishing Γ_{f^t} graphs.

Theorem 12 ([20]*, Theorem 2; [22]*, Theorem 1.2) *Let f^t, f'^t be Ω-stable flows without limit cycles on the surface S of genus g, and Γ_{f^t}, $\Gamma_{f'^t}$ be their n-vertexed and m-edged four-color graphs. Then the isomorphism of the graphs Γ_{f^t} and $\Gamma_{f'^t}$ is verified in time $O(n^{O(g)})$ for $g > 0$ and in time $O(n)$ if $g = 0$. The orientability of the surface S is calculated in time $O(n + m)$.*

For an arbitrary Ω-stable flow ϕ^t on the surface S, the equipped graph $\Upsilon^*_{\phi^t}$ is constructed similarly to the Morse-Smale flow graph using division into elementary regions with the exception, that the $\mathcal{M}$-vertices are equipped with a four-color graph $\Gamma_{\mathcal{M}}$, and a type 4 vertex is added, i.e. a $\mathcal{E}$-vertex corresponding to a region without fixed points and limiting cycles. Such a vertex is equipped with weights $+$, $-$ if the cycles in neighboring $\mathcal{L}$-regions are oriented consistently, unconsistemtly, respectively (see Fig. 9). .

Theorem 13 ([23]*, Theorem 5.3; [26]*, Theorem 4.1) *The Ω-stable flows ϕ^t and ϕ'^t are topologically equivalent iff their equipped graphs $\Upsilon^*_{\phi^t}$ and $\Upsilon^*_{\phi'^t}$ are isomorphic.*

The set of admissible equipped graphs is also distinguished and the following theorem is constructively proved.

Theorem 14 ([23]*, Theorem 5.9; [27]*, Theorem 1) *Every admissible equipped graph Υ^* corresponds to an Ω-stable flow $\phi^t\colon S \to S$ on a closed surface S, besides:*

(1) the Eulerian characteristic of the surface S is calculated by the formula

$$\chi(S) = \sum_{\mathcal{M}} (X_{\mathcal{M}} - Y_{\mathcal{M}}) + N_{\mathcal{A}},$$

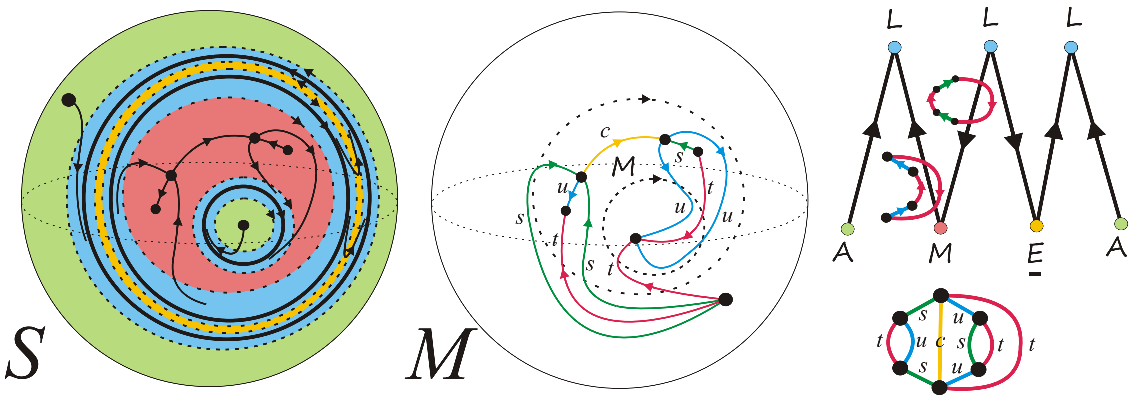

Figure 9: Ω-stable flow ϕ^t, its flow without limit cycles f^t, graph $\Upsilon^*_{\phi^t}$ flow ϕ^t

where $X_{\mathcal{M}}$ is the result of applying the formula (1) to the corresponding admissible four-color graph $\Gamma_{\mathcal{M}}$, $Y_{\mathcal{M}}$ is the number of edges incident to $\mathcal{M}$ and $N_{\mathcal{A}}$ is the number of $\mathcal{A}$-tops of Υ^*;

(2) The surface S is orientable if and only if every four-color graph equipping Υ^* has no cycles of odd length, and every $\mathcal{L}$ vertex has valence 2.

An efficient algorithm for distinguishing equipped graphs of Ω-stable streams is constructed.

Theorem 15 ([23]*, Theorem 5.10) *Let ϕ^t, ϕ'^t – Ω-stable flows on the surface S of genus g, and $\Upsilon^*_{\phi^t}$, $\Upsilon^*_{\phi'^t}$ – their n-verticed and m-edged equipped graphs. Then the isomorphism of graphs $\Upsilon^*_{\phi^t}$ and $\Upsilon^*_{\phi'^t}$ is verified in time $O(n^{O(g)})$ for $g > 0$ and in time $O(n)$ if $g = 0$. The orientability of the surface S is computed in time $O(n + m)$, the Eulerian characteristic can be computed in time $O(m^2)$.*

Another result of the chapter concerns the proof of finiteness of moduli of topological conjugacy of Ω-stable flows with connections of arbitrary length. For this purpose we consider a class of flows $f^t\colon S_g \to S_g$ of smoothness class C^2 generated by a gradient vector field of height function of vertical orientable surface S_g of genus $g > 0$. The disjoint set of such systems consists of a finite number of hyperbolic fixed points: one source, one sink, and $2g$ saddle points forming a chain of connections of length $2g - 1$ (see Fig. 10).

The moduli of such systems are the relations of eigenvalues of each pair of saddle points connected by bundles corresponding to invariant manifolds which do not participate in the connection of the given saddle points discovered by J. Palis [38]. The result of this section is the fact that no other moduli of such systems arise.

Theorem 16 ([29]*, Theorem 1.1) *The flow $f^t\colon S_g \to S_g$ has exactly $2g - 1$ moduli.*

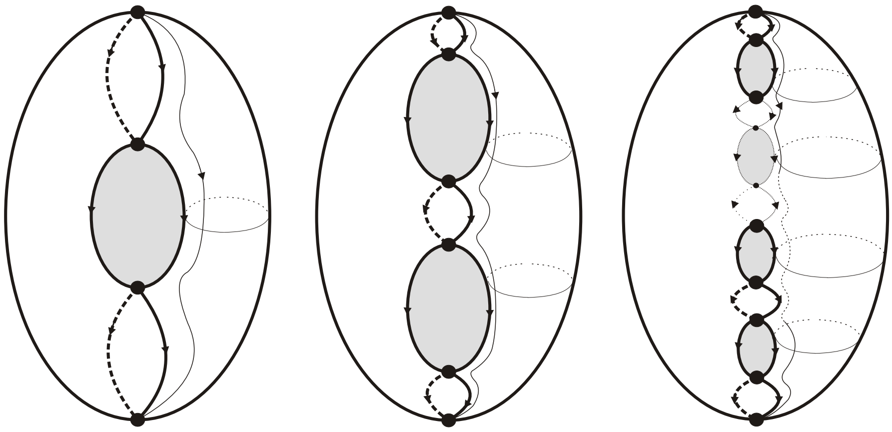

Figure 10: Examples of flows with chains of connections

2 Articles based on the research results

The results are presented in eight articles.

1. Kruglov V. E., Pochinka O. V. Classification with accuracy to topological conjugacy of Morse – Smale flows with finite number of moduli of stability on surfaces // News of higher educational institutions. Applied Nonlinear Dynamics. 2021. V. 29. No. 6. P. 835-850.

2. Kruglov V., Pochinka O. Criterion for the Topological Conjugacy of Multi-Dimensional Gradient-Like Flows with No Heteroclinic Intersections on a Sphere / Пер. с рус. // Journal of Mathematical Sciences. 2020. Vol. 250. P. 22-30.

3. Kruglov V., Malyshev D., Pochinka O., Shubin D. On Topological Classification of Gradient-like Flows on an n-sphere in the Sense of Topological Conjugacy // Regular and Chaotic Dynamics. 2020. Vol. 25. No. 6. P. 716-728.

4. V. Kruglov, O. Pochinka, G. Talanova. On functional moduli of surface flows // Proceedings of the International Geometry Center. 2020. Vol. 13. No. 1. P. 49-60.

5. Kruglov V., Malyshev D., Pochinka O. On Algorithms that Effectively Distinguish Gradient-Like Dynamics on Surfaces // Arnold Mathematical Journal. 2018. Vol. 4. No. 3-4. P. 483-504.

6. Vladislav Kruglov, Dmitry Malyshev, Olga Pochinka. Topological Classification of Ω-stable Flows on Surfaces by Means of Effectively Distinguishable Multigraphs // Discrete and Continuous Dynamical Systems. 2018. Vol. 38. No. 9. P. 4305-4327. doi

7. Kruglov V. E., Malyshev D. S., Pochinka O. V. Multicolour graph as a complete topological invariant for Ω-stable flows without periodic trajectories on surfaces // Sbornik: Mathematics. 2018. V. 209. No. 1. P. 100-126.

8. Kruglov V. E. On number of moduli for gradient surface height function flows // Zhurnal SVMO. 2018. V. 20. No. 4. P. 419-428.

3 Conclusion

In this dissertation three classes of surface flows with regular dynamics are considered: gradient-like flows, Morse-Smale flows and Ω-stable flows. The above classes were investigated in terms of the presence and number of moduli of topological conjugacy, and full equivalence invariants were found for them, and in the case of finiteness of the number of moduli, topological conjugacy invariants as well. For all classes the existence of a complete invariant which is an equipped graph is proved, the class of admissible equipped graphs with their subsequent realization by an appropriate thread is distinguished, and efficient algorithms for distinguishing isomorphism of equipped graphs are constructed.

Let us list the main results of the work.

- It is proved that gradient-like flows on surfaces are topologically conjugate if and only if they are topologically equivalent (Theorem 1).
- Effective algorithms for recognizing isomorphism of the following invariants of gradient-like flows on surfaces are constructed:

  - Peixoto graph (Theorem 2);

  - modified Peixoto graph (Theorem 3);

  - Wong graph (Theorem 4);

  - Fleitas graph (Theorem 5);

  - Oshemkov-Sharko tricolour graph (Theorem 6).
- There are found necessary and sufficient conditions that the Morse-Smale flow on the surface has a finite number of moduli of topological conjugacy (Theorem 8) and also a functional modulus is found for the Morse-Smale flow, which has stable and unstable limit cycles whose invariant manifolds intersect (Theorem 7).
- For Morse-Smale flows with a finite number of moduli a complete topological conjugacy invariant which is an equipped graph (Theorem 9) is constructed.
- For Ω-stable flows without limit cycles a complete topological equivalence invariant which is a four-color graph (Theorem 10) is constructed. The class of admissible equipped graphs, for each of which a flow in the class under consideration is constructed (Theorem 11), and an effective algorithm to distinguish isomorphism of such graphs (Theorem 12) is also constructed.

- For Ω-stable flows in the general case a complete invariant of topological equivalence which is an equipped graph (Theorem 13) is constructed. The class of admissible equipped graphs, for each of which a flow in the class under consideration is constructed (Theorem 14), and an effective algorithm to distinguish isomorphism of such graphs (Theorem 15) is also constructed.
- It was found that flows with arbitrarily long chain of connection, in contrast to diffeomorphisms, have a finite number of moduli of topological conjugacy (Theorem 16).

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